**Circle theorems**

 **A LEVEL LINKS**

 **Scheme of work:** 2b. Circles – equation of a circle, geometric problems on a grid

Key points

* A chord is a straight line joining two points on the circumference of a circle.
So AB is a chord.



* A tangent is a straight line that touches the circumference of a circle at only one point.
The angle between a tangent and the radius is 90°.



* Two tangents on a circle that meet at a point outside the circle are equal in length.
So AC = BC.



* The angle in a semicircle is a right angle.
So angle ABC = 90°.



* When two angles are subtended by the same arc, the angle at the centre of a circle is twice the angle at the circumference.
So angle AOB = 2 × angle ACB.
* Angles subtended by the same arc at the circumference are equal. This means that angles in the same segment are equal.
So angle ACB = angle ADB and
angle CAD = angle CBD.



* A cyclic quadrilateral is a quadrilateral with all four vertices on the circumference of a circle.
Opposite angles in a cyclic quadrilateral total 180°.
So *x* + *y* = 180° and *p* + *q* = 180°.



* The angle between a tangent and chord is equal to the angle in the alternate segment, this is known as the alternate segment theorem.
So angle BAT = angle ACB.

Examples

**Example 1** Work out the size of each angle
marked with a letter.
Give reasons for your answers.

|  |  |
| --- | --- |
| Angle *a* = 360° − 92° = 268° as the angles in a full turn total 360°.Angle *b* = 268° ÷ 2 = 134°as when two angles are subtended by the same arc, the angle at the centre of a circle is twice the angle at the circumference. | **1** The angles in a full turn total 360°.**2** Angles *a* and *b* are subtended by the same arc, so angle *b* is half of angle *a*. |

**Example 2** Work out the size of the angles in the triangle.
 Give reasons for your answers.

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| --- | --- |
| Angles are 90°, 2*c* and *c*.90° + 2*c* + *c* = 180° 90° + 3*c* = 180° 3*c* = 90° *c* = 30° 2*c* = 60°The angles are 30°, 60° and 90° as the angle in a semi-circle is a right angle and the angles in a triangle total 180°. | **1** The angle in a semicircle is a right angle. **2** Angles in a triangle total 180°.**3** Simplify and solve the equation. |



**Example 3** Work out the size of each angle marked with a letter.
 Give reasons for your answers.

|  |  |
| --- | --- |
| Angle *d* = 55° as angles subtended by the same arc are equal.Angle *e* = 28° as angles subtended by the same arc are equal. | **1** Angles subtended by the same arc are equal so angle 55° and angle *d* are equal.**2** Angles subtended by the same arc are equal so angle 28° and angle *e* are equal. |



**Example 4** Work out the size of each angle marked with a letter.
 Give reasons for your answers.

|  |  |
| --- | --- |
| Angle *f* = 180° − 94° = 86° as opposite angles in a cyclic quadrilateral total 180°. | **1** Opposite angles in a cyclic quadrilateral total 180° so angle 94° and angle *f* total 180°.*(continued on next page)* |
| Angle *g* = 180° − 86° = 84° as angles on a straight line total 180°.Angle *h* = angle *f* = 86° as angles subtended by the same arc are equal. | **2** Angles on a straight line total 180° so angle *f* and angle *g* total 180°.**3** Angles subtended by the same arc are equal so angle *f* and angle *h* are equal. |



**Example 5** Work out the size of each angle marked with a letter.
 Give reasons for your answers.

|  |  |
| --- | --- |
| Angle *i* = 53° because of the alternate segment theorem.Angle *j* = 53° because it is the alternate angle to 53°.Angle *k* = 180° − 53° − 53° = 74° as angles in a triangle total 180°. | **1** The angle between a tangent and chord is equal to the angle in the alternate segment.**2** As there are two parallel lines, angle 53° is equal to angle *j* because they are alternate angles.**3** The angles in a triangle total 180°, so *i* + *j* + *k* = 180°. |



**Example 6** XZ and YZ are two tangents to a circle with centre O.
 Prove that triangles XZO and YZO are congruent.

|  |  |
| --- | --- |
| Angle OXZ = 90° and angle OYZ = 90° as the angles in a semicircle are right angles.OZ is a common line and is the hypotenuse in both triangles.OX = OY as they are radii of the same circle.So triangles XZO and YZO are congruent, RHS. | For two triangles to be congruent you need to show one of the following.* All three corresponding sides are equal (SSS).
* Two corresponding sides and the included angle are equal (SAS).
* One side and two corresponding angles are equal (ASA).
* A right angle, hypotenuse and a shorter side are equal (RHS).
 |

Practice

**1** Work out the size of each angle marked with a letter.

**** Give reasons for your answers.

** a b**

****

 **c d**

A

 **e**

**2** Work out the size of each angle marked with a letter.
 Give reasons for your answers.

** a b**



 **c**

**Hint**

The reflex angle at point O and angle *g* are subtended by the same arc. So the reflex angle is twice the size of angle *g*.

 **d**

**Hint**

Angle 18° and angle *h* are subtended by the same arc.

**3** Work out the size of each angle marked with a letter.
 Give reasons for your answers.

 **a b**

**Hint**

One of the angles is in a semicircle.

****

 **c d**

**4** Work out the size of each angle marked with a letter.
 Give reasons for your answers.

 **a**

**Hint**

An exterior angle of a cyclic quadrilateral is equal to the opposite interior angle.

** b c**

 **d**

**Hint**

One of the angles is in a semicircle.

Extend

**5** Prove the alternate segment theorem.

Answers

**1 a** *a* = 112°, angle OAP = angle OBP = 90° and angles in a quadrilateral total 360°.

 **b** *b* = 66°, triangle OAB is isosceles, Angle OAP = 90° as AP is tangent to the circle.

 **c** *c* = 126°, triangle OAB is isosceles.
 *d* = 63°, Angle OBP = 90° as BP is tangent to the circle.

 **d** *e* = 44°, the triangle is isosceles, so angles *e* and angle OBA are equal. The angle OBP = 90° as BP is tangent to the circle.
 *f* = 92°, the triangle is isosceles.

 **e** *g* = 62°, triangle ABP is isosceles as AP and BP are both tangents to the circle.
 *h* = 28°, the angle OBP = 90°.

**2 a** *a* = 130°, angles in a full turn total 360°.
 *b* = 65°, the angle at the centre of a circle is twice the angle at the circumference.
 *c* = 115°, opposite angles in a cyclic quadrilateral total 180°.

 **b** *d* = 36°, isosceles triangle.
 *e* = 108°, angles in a triangle total 180°.
 *f* = 54°, angle in a semicircle is 90°.

 **c** *g* = 127°, angles at a full turn total 360°, the angle at the centre of a circle is twice the angle at the circumference.

 **d** *h* = 36°, the angle at the centre of a circle is twice the angle at the circumference.

**3 a** *a* = 25°, angles in the same segment are equal.
 *b* = 45°, angles in the same segment are equal.

 **b** *c* = 44°, angles in the same segment are equal.
 *d* = 46°, the angle in a semicircle is 90° and the angles in a triangle total 180°.

 **c** *e* = 48°, the angle at the centre of a circle is twice the angle at the circumference.
 *f* = 48°, angles in the same segment are equal.

 **d** *g* = 100°, angles at a full turn total 360°, the angle at the centre of a circle is twice the angle at the circumference.
 *h* = 100°, angles in the same segment are equal.

**4 a** *a* = 75°, opposite angles in a cyclic quadrilateral total 180°.
 *b* = 105°, angles on a straight line total 180°.
 *c* = 94°, opposite angles in a cyclic quadrilateral total 180°.

 **b** *d* = 92°, opposite angles in a cyclic quadrilateral total 180°.
 *e* = 88°, angles on a straight line total 180°.
 *f* = 92°, angles in the same segment are equal.

 **c** *h* = 80°, alternate segment theorem.

 **d** *g* = 35°, alternate segment theorem and the angle in a semicircle is 90°.

**5** Angle BAT = *x*.

 Angle OAB = 90° − *x* because the angle between the tangent and the radius is 90°.

 OA = OB because radii are equal.

 Angle OAB = angle OBA because the base of isosceles triangles are equal.

 Angle AOB = 180° − (90° − *x*) − (90° − *x*) = 2*x* because angles in a triangle total 180°.

 Angle ACB = 2*x* ÷ 2 = *x* because the angle at the centre is twice the angle at the circumference.